

# PROBLEMS IN ADDITIVE NUMBER THEORY, III: THEMATIC SEMINARS AT THE CENTRE DE RECERCA MATEMÀTICA

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ABSTRACT. This is a survey of open problems in different parts of combinatorial and additive number theory.

## 1. WHAT SETS ARE SUMSETS?

Let  $\mathbf{N}, \mathbf{N}_0, \mathbf{Z}$ , and  $\mathbf{Z}^d$  denote, respectively, the sets of positive integers, nonnegative integers, integers, and  $d$ -dimensional integral lattice points. Let  $\mathbf{G}$  denote an arbitrary abelian group and let  $\mathbf{X}$  denote an arbitrary abelian semigroup, written additively. Let  $|S|$  denote the cardinality of the set  $S$ . For any sets  $A$  and  $B$ , we write  $A \sim B$  if their symmetric difference is finite, that is, if  $|(A \setminus B) \cup (B \setminus A)| < \infty$ .

**Definition 1.** *Let  $A$  and  $B$  be nonempty subsets of an additive abelian semigroup  $\mathbf{X}$ . The most important definition in additive number theory is the sumset  $A + B$  of the sets  $A$  and  $B$ :*

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let  $A + B = \emptyset$  if  $A = \emptyset$  or  $B = \emptyset$ . If  $h \geq 3$  and  $A_1, \dots, A_h$  are subsets of  $\mathbf{X}$ , then we construct the sumset  $A_1 + \dots + A_h$  inductively as follows:

$$\begin{aligned} A_1 + \dots + A_{h-1} + A_h &= (A_1 + \dots + A_{h-1}) + A_h \\ &= \{a_1 + \dots + a_h : a_i \in A_i \text{ for all } i = 1, \dots, h\}. \end{aligned}$$

If  $A_1 = A_2 = \dots = A_h = A$ , then

$$hA = \underbrace{A + \dots + A}_{h \text{ times}}$$

is called the  $h$ -fold sumset of  $A$ .

**Definition 2.** *Let  $A$  be a nonempty subset of an additive abelian semigroup  $\mathbf{X}$ . The set  $A$  is called a basis of order  $h$  for  $\mathbf{X}$  if  $hA = \mathbf{X}$ . The set  $A$  is called an asymptotic basis of order  $h$  for  $\mathbf{X}$  if  $hA \sim \mathbf{X}$ .*

A basic problem is: What sets are sumsets? More precisely,

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**Problem 1.** *Given a set  $S \subseteq \mathbf{X}$ , do there exist sets  $A_1, \dots, A_h$  of integers such that  $A_1 + \dots + A_h = S$  or  $A_1 + \dots + A_h \sim S$ ?*

**Problem 2.** *Given a set  $S \subseteq \mathbf{X}$ , does there exist a set  $A$  of integers such that  $hA = S$  or  $hA \sim S$ ?*

These problems are particularly important in the classical cases  $\mathbf{X} = \mathbf{N}_0$ ,  $\mathbf{X} = \mathbf{Z}$ , and  $\mathbf{X} = \mathbf{Z}^d$ .

## 2. DESCRIBING THE STRUCTURE OF $hA$ AS $h \rightarrow \infty$

**Definition 3.** *Let  $A$  be a set of nonnegative integers. The counting function  $A(x)$  of the set  $A$  counts the number of positive elements of  $A$  not exceeding  $x$ , that is,  $A(x) = |A \cap [1, x]|$ . The lower asymptotic density of  $A$  is*

$$d_L(A) = \liminf_{x \rightarrow \infty} \frac{A(x)}{x}.$$

**Theorem 1** (Nash-Nathanson [20]). *If  $A$  is a set of nonnegative integers with  $\gcd(A) = 1$  such that the sumset  $h_0 A$  has positive lower asymptotic density for some positive integer  $h_0$ , then there exists a number  $h \in \mathbf{N}$  such that  $hA \sim \mathbf{N}_0$ .*

Equivalently, a set  $A$  of nonnegative integers is an asymptotic basis of finite order if and only if  $\gcd(A) = 1$  and  $d_L(h_0 A) > 0$  for some positive integer  $h_0$ .

Let  $A$  be a set of nonnegative integers with  $0 \in A$  and  $\gcd(A) = 1$ . We have the increasing sequence of sets

$$A \subseteq 2A \subseteq \dots \subseteq hA \subseteq (h+1)A \subseteq \dots.$$

If some sumset has positive lower asymptotic density, then this sequence becomes eventually constant and equal to  $A \setminus \mathcal{F}$  for some finite set  $\mathcal{F}$  of integers. An important unsolved question is the following:

**Problem 3.** *Suppose that  $d_L(hA) = 0$  for all  $h \geq 1$ . Describe the evolution of the structure of the sumset  $hA$  as  $h \rightarrow \infty$ .*

## 3. REPRESENTATION FUNCTIONS

Let  $\mathcal{A} = (A_1, \dots, A_h)$  be an  $h$ -tuple of subsets of an additive abelian semigroup  $\mathbf{X}$ . We want to count the number  $R_{\mathcal{A}}(x)$  of representations of an element  $x \in \mathbf{X}$  in the form  $x = a_1 + \dots + a_h$  with  $a_i \in A_i$  for  $i = 1, \dots, h$ . We discuss here only the special case when  $A_i = A$  for  $i = 1, \dots, h$ . We shall consider two different representation functions.

**Definition 4.** *The ordered representation function of order  $h$  for the set  $A$  is the function  $R_{A,h} : \mathbf{X} \rightarrow \mathbf{N}_0 \cup \{\infty\}$  defined by*

$$R_{A,h}(x) = |\{(a_1, \dots, a_h) \in A^h : x = a_1 + \dots + a_h\}|.$$

**Definition 5.** *Let  $(a_1, \dots, a_h) \in A^h$  and  $(a'_1, \dots, a'_h) \in A^h$  be  $h$ -tuples that represent  $x$ , that is,  $x = a_1 + \dots + a_h = a'_1 + \dots + a'_h$ . These representations are called equivalent if there is a permutation  $\sigma$  of the set  $\{1, 2, \dots, h\}$  such that  $a'_i = a_{\sigma(i)}$  for  $i = 1, 2, \dots, h$ . The unordered representation function  $r_{A,h}(x)$  of order  $h$  counts the number of equivalence classes of representations of  $x$ .*

If  $\mathbf{X}$  is a linearly ordered semigroup such as  $\mathbf{N}_0$  or  $\mathbf{Z}$ , then we can write

$$r_{A,h}(x) = |\{(a_1, \dots, a_h) \in A^h : x = a_1 + \dots + a_h \text{ and } a_1 \leq a_2 \leq \dots \leq a_h\}|$$

Let  $\mathcal{F}(\mathbf{X})$  denote the set of all functions  $f : \mathbf{X} \rightarrow \mathbf{N}_0 \cup \{\infty\}$ , and let

$$\mathcal{R}_h(\mathbf{X}) = \{r_{A,h} : A \subseteq \mathbf{X}\}$$

denote the set of all unordered representation functions order  $h$  of subsets of  $\mathbf{X}$ . Then  $\mathcal{R}_h(\mathbf{X}) \subseteq \mathcal{F}(\mathbf{X})$ . A simple question is:

**Problem 4.** *What functions are representation functions?*

This problem seems hopelessly difficult at this time. We consider the special case of representation functions of asymptotic bases. Define the function space

$$\mathcal{F}^{(0)}(\mathbf{X}) = \{f : \mathbf{X} \rightarrow \mathbf{N}_0 \cup \{\infty\} : |f^{-1}(0)| < \infty\}.$$

This is the space of functions on  $\mathbf{X}$  with only finitely many zeros. Let

$$\mathcal{R}_h^{(0)}(\mathbf{X}) = \{r_{A,h} : A \subseteq \mathbf{X} \text{ and } hA \sim \mathbf{X}\}$$

be the set of representation functions of asymptotic bases of order  $h$  for  $\mathbf{X}$ . Then

$$\mathcal{R}_h^{(0)}(\mathbf{X}) \subseteq \mathcal{F}^{(0)}(\mathbf{X}).$$

**Problem 5.** *What functions in  $\mathcal{F}_0(\mathbf{X})$  are representation functions of asymptotic bases? Equivalently, what are necessary and sufficient conditions for a function on  $\mathbf{X}$  with only finitely many zeros to be the representation function of an additive basis?*

For the group  $\mathbf{Z}$  of integers there is the following amazing result.

**Theorem 2** (Nathanson [24, 25, 26]). *For every integer  $h \geq 2$  and for every function  $\mathcal{F}_0(\mathbf{Z})$  there exists a set  $A \subseteq \mathbf{Z}$  such that  $f = r_{A,h}$ . Equivalently,*

$$\mathcal{F}_0(\mathbf{Z}) = \mathcal{R}_2^{(0)}(\mathbf{Z}) = \mathcal{R}_3^{(0)}(\mathbf{Z}) = \dots = \mathcal{R}_h^{(0)}(\mathbf{Z}) = \dots.$$

The classification of arbitrary representation functions for the integers is still open.

**Problem 6.** *What functions in  $\mathcal{F}(\mathbf{Z})$  are representation functions of sumsets of order  $h$ ?*

Very little is known about representation functions of sums of sets of nonnegative integers, even for sets that are asymptotic bases. G. A. Dirac gave an elegant proof of the following beautiful result.

**Theorem 3** (Dirac [4]). *If  $A$  is an asymptotic basis of order 2 for  $\mathbf{N}_0$ , then the unordered representation function  $r_{A,2}$  is not eventually constant.*

*Proof.* The generating function  $G_A(z) = \sum_{a \in A} z^a$  converges in the open unit disc and diverges as  $z \rightarrow 1^-$ . We have  $G_A^2(z) = \sum_{n=0}^{\infty} R_A(n)z^n$  and

$$\frac{G_A^2(z) + G_A(z^2)}{2} = \sum_{n=0}^{\infty} r_{A,2}(n)z^n.$$

If  $r_{A,2}(n) = c$  for all  $n \geq n_0$ , then there is a polynomial  $P(z)$  such that

$$\frac{G_A^2(z) + G_A(z^2)}{2} = P(z) + \frac{c}{1-z}.$$

Let  $0 < x < 1$  and  $z = -x$ . Then

$$G_A(x^2) \leq G_A^2(-x) + G_A(x^2) = 2P(-x) + \frac{2c}{1+x}.$$

As  $x \rightarrow 1^-$ , the right side approaches  $2P(-1) + c$  while the left side diverges to  $\infty$ , which is absurd.  $\square$

One of the most famous and tantalizing unsolved problems in additive number theory is the following:

**Problem 7** (Erdős-Turán [5]). *Let  $h \geq 2$ . Prove that if  $A$  is an asymptotic basis of order  $h$  for the nonnegative integers, then the representation function  $r_{A,h}$  must be unbounded.*

#### 4. SETS WITH MORE SUMS THAN DIFFERENCES

Let  $A$  be a finite set of integers. We define the sumset

$$A + A = \{a + a' : a, a' \in A\}$$

and the difference set

$$A - A = \{a - a' : a, a' \in A\}.$$

Since  $2 + 3 = 3 + 2$  but  $2 - 3 \neq 3 - 2$ , that is, since addition is commutative but subtraction is not commutative, it would be reasonable to conjecture that  $|A + A| \leq |A - A|$  for every finite set  $A$  of integers. In some special cases, for example, if  $A$  is an arithmetic progression or if  $A$  is symmetric (that is, if  $A = c - A$  for some  $c \in \mathbf{Z}$ ), then  $|A + A| = |A - A|$ . The “conjecture,” however, is false. The simplest counterexample is the set

$$A^* = \{0, 2, 3, 4, 7, 11, 12, 14\}$$

for which  $|A^* + A^*| = 26 > 25 = |A^* - A^*|$ . This set is “almost” symmetric. We have  $A^* = A' \cup \{4\}$  where  $A' = \{0, 2\} \cup \{3, 7, 11\} \cup \{12, 14\}$  satisfies  $A' = 14 - A'$ . A set with more sums than differences is called an *MSTD set*. Nathanson [28] showed that if  $k \geq 3$  and  $A' = \{0, 2\} \cup \{3, 7, 11, 15, 19, 23, \dots, 4k - 1\} \cup \{4k, 4k + 2\}$  and  $A^* = A' \cup \{4\}$  then  $A^*$  is an MSTD set. Many other examples of MSTD sets have been constructed by Hegarty [12].

Finite sets  $A$  of integers with the property that  $|A + A| > |A - A|$  are extremely interesting, since a sumset really should have more elements than the corresponding difference set.

**Conjecture 1.**  $|A + A| \leq |A - A|$  for almost all finite sets  $A$ .

Martin and O’Bryant [19] studied the uniform probability measure on the set of all subsets of  $\{1, \dots, N\}$ , that is, they assigned to each subset the probability  $2^{-N}$ . Counting sets in this way, they calculated that the average cardinality of a sumset was

$$\frac{1}{2^N} \sum_{A \subseteq \{1, \dots, N\}} |A + A| = 2N - 11$$

and the average cardinality of a difference set was

$$\frac{1}{2^N} \sum_{A \subseteq \{1, \dots, N\}} |A - A| = 2N - 7.$$

Thus, on average, a difference set contains four more elements than the sumset. However, they also proved the following result.

**Theorem 4** (Martin-O'Bryant [19]). *With the uniform probability measure, there exists a  $\delta > 0$  such that*

$$\text{Prob}(|A + A| > |A - A| : A \subseteq \{1, \dots, N\}) > \delta$$

for all  $N \geq N_0$

Thus, choosing a uniform probability measure, it is not true that almost all sets have more differences than sums. Of course, with the uniform probability distribution most subsets of the interval  $\{1, \dots, N\}$  are large and satisfy  $|A + A| = |A - A| = 2N - 1$ . This skews the calculation.

Using a binomial probability distribution, Hegarty and Miller obtained a very different result.

**Theorem 5** (Hegarty-Miller [13]). *Let  $p : \mathbf{N} \rightarrow (0, 1)$  be a function such that  $\lim_{N \rightarrow \infty} p(N) = 0$  and  $\lim_{N \rightarrow \infty} Np(N) = \infty$ . Define the function  $q : \mathbf{N} \rightarrow (0, 1)$  by  $q(N) = 1 - p(N)$ . Consider the binomial probability distribution with parameter  $p(N)$  on the space of all subsets of  $\{1, \dots, N\}$ , so that a subset of size  $k$  has probability  $p(N)^k q(N)^{N-k}$ . Then*

$$\lim_{N \rightarrow \infty} \text{Prob}(|A + A| > |A - A| : A \subseteq \{1, \dots, N\}) = 0.$$

These theorems seem to contradict each other, but they do not because they use different probability measures.

**Problem 8.** *A difficult and subtle problem is to decide what is the appropriate method of counting (or, equivalently, the appropriate probability measure) to apply to MSTD sets.*

## 5. COMPARATIVE THEORY OF LINEAR FORMS

Let  $f(x_1, \dots, x_m)$  be an integer-valued function on the integers and let  $A$  be a set of integers. We define the set

$$f(A) = \{f(a_1, \dots, a_m) : a_1, \dots, a_m \in A\}.$$

In particular, if  $f(x_1, x_2) = u_1x_1 + u_2x_2$  is a linear form with nonzero integer coefficients, then

$$f(A) = \{u_1a_1 + u_2a_2 : a_1, a_2 \in A\}.$$

For example, if  $s(x_1, x_2) = x_1 + x_2$ , then  $s(A)$  is the sumset  $A + A$ . If  $d(x_1, x_2) = x_1 - x_2$ , then  $d(A)$  is the difference set  $A - A$ .

The binary linear forms  $f(x_1, x_2) = u_1x_1 + u_2x_2$  and  $g(x_1, x_2) = v_1x_1 + v_2x_2$  are *related* if

$$(v_1, v_2) = (u_2, u_1), \text{ or}$$

$$(v_1, v_2) = (du_1, du_2) \text{ for some integer } d, \text{ or}$$

$$(v_1, v_2) = (u_1/d, u_2/d) \text{ for some integer } d \text{ that divides } u_1 \text{ and } u_2.$$

The binary linear forms  $f(x_1, x_2) = u_1x_1 + u_2x_2$  and  $g(x_1, x_2) = v_1x_1 + v_2x_2$  are *equivalent* if there is a finite sequence of binary linear forms  $f_0, f_1, \dots, f_k$  such that  $f = f_0$ ,  $g = f_k$ , and  $f_{i-1}$  is related to  $f_i$  for all  $i = 1, \dots, k$ . If  $f$  and  $g$  are equivalent forms, then  $|f(A)| = |g(A)|$  for every finite set  $A$ .

The binary linear form  $f(x_1, x_2) = u_1x_1 + u_2x_2$  is *normalized* if  $\gcd(u_1, u_2) = 1$  and  $u_1 \geq |u_2| \geq 1$ . Every binary linear form is equivalent to a unique normalized form.

The following is the basic result in the comparative theory of linear forms.

**Theorem 6** (Nathanson-O'Bryant-Orosz-Ruzsa-Silva [32]). *Let  $f(x_1, x_2) = u_1x_1 + u_2x_2$  and  $g(x_1, x_2) = v_1x_1 + v_2x_2$  be distinct normalized linear forms. There exist sets  $A$  and  $A'$  of integers such that  $|f(A)| < |g(A)|$  and  $|f(A')| > |g(A')|$ .*

Consider now linear forms in more than two variables.

**Problem 9.** *Let  $f(x_1, \dots, x_m) = u_1x_1 + \dots + u_mx_m$  and  $g(x_1, \dots, x_m) = v_1x_1 + \dots + v_mx_m$  be linear forms with nonzero integer coefficients in  $m \geq 3$  variables. Suppose that  $\gcd(u_1, \dots, u_m) = \gcd(v_1, \dots, v_m) = 1$  and that  $g$  cannot be obtained from  $f$  by some permutation of the coefficients or by multiplication by  $-1$ . Does there exist a finite set  $A$  of integers such that  $|f(A)| > |g(A)|$ ?*

**Problem 10.** *Let  $f(x_1, \dots, x_m)$  and  $g(x_1, \dots, x_m)$  be polynomials with integer coefficients in  $m \geq 2$  variables. Under what conditions does there exist a finite set  $A$  of integers such that  $|f(A)| > |g(A)|$ ?*

**Definition 6.** *Let  $f$  be an integer-valued function defined on  $\mathbf{Z}$ . Define  $N_f(k) = \min \{|f(A)| : A \subseteq \mathbf{Z} \text{ and } |A| = k\}$ .*

**Problem 11.** *Let  $f$  be an integer-valued function defined on  $\mathbf{Z}$ , for example, a linear form or a polynomial with integer coefficients. Determine  $N_f(k)$  and describe the structure of the minimizing sets.*

**Theorem 7** (Bukh [2]). *Let  $f(x_1, \dots, x_m) = u_1x_1 + \dots + u_mx_m$  be a linear form with nonzero integer coefficients in  $m \geq 2$  variables. If  $\gcd(u_1, \dots, u_m) = 1$ , then  $N_f(k) = \left( |u_1| + \dots + |u_m| \right)k - o(k)$*

**Theorem 8.** *If  $f(x_1, x_2) = x_1 + x_2$  then  $N_f(k) = 2k - 1$  and the minimizing sets are finite arithmetic progressions. Equivalently, if  $A$  is a finite set of integers, then  $|A + A| \geq 2|A| - 1$  and  $|A + A| = 2|A| - 1$  if and only if  $A$  is an arithmetic progression.*

An *affine transform* of a set  $A$  of real numbers is a set obtained from  $A$  by a sequence of translations and dilations.

**Theorem 9** (Cilleruelo-Silva-Vinuesa [15]). *If  $f(x_1, x_2) = x_1 + 2x_2$ , then  $N_f(k) = 3k - 2$ . Moreover,  $|f(A)| = 3|A| - 2$  if and only if  $A$  is an arithmetic progression.*

*If  $f(x_1, x_2) = x_1 + 3x_2$ , then  $N_f(k) = 4k - 4$ . Moreover,  $|f(A)| = 4|A| - 4$  if and only if  $A$  is  $\{0, 1, 3\}$  or  $\{0, 1, 4\}$  or  $\{0, 3, 6, \dots, 3\ell - 3\} \cup \{1, 4, 7, \dots, 3\ell - 2\}$ , or an affine transform of one of these sets.*

**Definition 7.** *Let  $\mathcal{U} = (u_1, \dots, u_m)$  be a sequence of positive integers. A subsequence sum of  $\mathcal{U}$  is a nonnegative integer of the form  $\sum_{i \in I} u_i$ , where  $I$  is a subset of  $\{1, \dots, m\}$ . Let  $S(\mathcal{U}) = \{\sum_{i \in I} u_i : I \subseteq \{1, \dots, m\}\}$  denote the set of all subsequence sums of the sequence  $\mathcal{U}$ .*

A subsequence sum is 0 if and only if  $I = \emptyset$ . If  $\mathcal{U} = u_1 + \dots + u_m$ , then  $S(\mathcal{U}) \subseteq [0, \mathcal{U}]$ .

**Definition 8.** *The sequence  $\mathcal{U}$  is called complete if  $S(\mathcal{U}) = [0, \mathcal{U}]$ .*

For example, the sequences  $(1, 1)$  and  $(1, 2)$  are complete but  $(1, 3)$  is not complete.

**Theorem 10** (Nathanson [29]). *Let  $\mathcal{U} = (u_1, \dots, u_m)$  be a complete sequence of positive integers, and let  $f(x_1, \dots, x_m) = u_1x_1 + \dots + u_mx_m$  be the associated linear form. If  $U = u_1 + \dots + u_m$ , then  $N_f(k) = Uk - U + 1$  for all positive integers  $k$ . Moreover,  $|A| = k$  and  $|f(A)| = N_f(k)$  if and only if  $A$  is an arithmetic progression of length  $k$ .*

There is also the dual problem of describing the finite sets of integers whose images under linear maps are large.

**Definition 9.** *Let  $f$  be an integer-valued function defined on  $\mathbf{Z}$ . Define  $M_f(k) = \max \{|f(A)| : A \subseteq \mathbf{Z}, |A| = k\}$ .*

**Problem 12.** *Let  $f$  be an integer-valued function in  $m$  variables defined on  $\mathbf{Z}$ . Determine  $M_f(k)$  and describe the structure of the maximizing sets. For what functions  $f$  is  $M_f(k) < k^m$ ?*

## 6. THE FUNDAMENTAL THEOREM OF ADDITIVE NUMBER THEORY

Let  $A = \{a_0 < a_1 < \dots < a_{k-1}\}$  be a finite set of integers. Consider the shifted set  $A' = A - \{a_0\} = \{0 < a_1 - a_0 < \dots < a_{k-1} - a_0\}$ . Let  $d = \gcd\{a_i - a_0 : i = 1, \dots, k-1\}$ , and construct the set

$$A^{(N)} = \frac{1}{d} * A' = \left\{ 0 < \frac{a_1 - a_0}{d} < \dots < \frac{a_{k-1} - a_0}{d} \right\}.$$

If  $hA$  is the  $h$ -fold sumset of  $A$ , then  $hA' = hA - \{ha_0\}$  and

$$hA^{(N)} = \frac{1}{d} * hA' = \frac{1}{d} * (hA - \{ha_0\}).$$

In particular,  $|hA| = |hA^{(N)}|$ .

The set  $A^{(N)}$  is called the *normalized* form of the set  $A$ . In general, a finite set  $A$  of integers is normalized if  $A = \{0\}$ , or if  $|A| \geq 2$ ,  $\min(A) = 0$ , and  $\gcd(A) = 1$ .

The following result is often called the Fundamental Theorem of Additive Number Theory.

**Theorem 11** (Nathanson [21]). *Let  $A = \{0 < a_1 < \dots < a_{k-1}\}$  be a normalized finite set of integers. There exist a positive integer  $h_0$ , nonnegative integers  $C$  and  $D$ , and finite sets  $\mathcal{C} \subseteq [0, C-2]$  and  $\mathcal{D} \subseteq [0, D-2]$  such that*

$$hA = \mathcal{C} \cup [C, ha_{k-1} - D] \cup (\{ha_{k-1}\} - \mathcal{D})$$

for all  $h \geq h_0$ .

If  $A$  is a set of nonnegative integers that contains 0, then

$$A \subseteq 2A \subseteq \dots \subseteq hA \subseteq (h+1)A \subseteq \dots$$

for all  $h \geq 1$ , and the set

$$\Sigma(A) = \bigcup_{h=1}^{\infty} hA$$

is the additive subsemigroup of the nonnegative integers generated by the set  $A$ . The fundamental theorem implies that if  $\gcd(A) = 1$ , then

$$\Sigma(A) = \mathcal{C} \cup [C, \infty).$$

The number  $C - 1$  is the largest integer that cannot be represented as a nonnegative integral linear combination of  $a_1, \dots, a_{k-1}$ . This is called the *Frobenius number* of the set  $A$ . Note that  $D - 1$  is the Frobenius number of the symmetric normalized set

$$A^\# = \{a_{k-1}\} - A = \{0 < a_{k-1} - a_{k-2} < \dots < a_{k-1} - a_1 < a_{k-1}\}.$$

Also, since  $\Sigma(A)$  is a semigroup, it follows that if  $u$  and  $v$  are nonnegative integers with  $u + v = C - 1$ , then either  $u \notin \Sigma(A)$  or  $v \notin \Sigma(A)$ . Therefore,

$$|\mathbf{N}_0 \setminus \Sigma(A)| = |[0, C - 1] \setminus \mathcal{C}| \geq \frac{C}{2}.$$

**Problem 13.** *Let  $A$  be a normalized finite set of nonnegative integers. Compute  $\mathbf{N}_0 \setminus \Sigma(A)$ , that is, the set of numbers that cannot be represented as nonnegative integral linear combinations of the elements of  $A$ .*

## 7. THIN ASYMPTOTIC BASES

Let  $A$  be an infinite set of nonnegative integers that is an asymptotic basis of order  $h$ . There is a nonnegative integer  $n_0$  such that, if  $n_0 \leq n \leq x$ , then there exist  $a_1, \dots, a_h \in A$  with  $n = a_1 + \dots + a_h$  and  $0 \leq a_i \leq n \leq x$  for  $i = 1, \dots, h$ . Denote by  $A(x)$  the counting function of the set  $A$ . Since the interval  $[n_0, x]$  contains at least  $x - n_0$  nonnegative integers, it follows that  $(A(x) + 1)^h \geq x - n_0$  and so

$$A(x) \gg x^{1/h}$$

for every asymptotic basis  $A$  of order  $h$ .

**Definition 10.** *An asymptotic basis  $A$  of order  $h$  is called thin if  $A(x) \ll x^{1/h}$ .*

Thin asymptotic bases exist, and the first explicit examples were constructed independently by Chartrovsky, Raikov, and Stöhr in the 1930s. Thin bases of order  $h$  have the property that their counting functions have order of magnitude  $x^{1/h}$ . Cassels constructed a family of bases of order  $h$  whose counting functions are asymptotic to  $\lambda x^{1/h}$  for some positive real number  $\lambda$ .

**Theorem 12** (Cassels [3]). *For every  $h \geq 2$ , there exist strictly increasing sequences  $A = \{a_n\}_{n=1}^\infty$  of nonnegative integers such that  $hA = \mathbf{N}_0$  and  $a_n = \lambda n^h + O(n^{h-1})$  for some  $\lambda > 0$ .*

**Problem 14** (Cassels [3]). *Let  $h \geq 2$ . Does there exist an asymptotic basis  $A = \{a_n\}_{n=1}^\infty$  of order  $h$  such that  $a_n = \lambda n^h + o(n^{h-1})$  for some  $\lambda > 0$ ?*

**Definition 11.** *A positive real number  $\lambda$  will be called an additive eigenvalue of order  $h$  if there exists an asymptotic basis  $A = \{a_n\}_{n=1}^\infty$  of order  $h$  such that*

$$a_n \sim \lambda n^h.$$

*We define the additive spectrum  $\Lambda_h$  as the set of all additive eigenvalues of order  $h$ .*

**Theorem 13** (Nathanson [31]). *For every integer  $h \geq 2$ , there is a number  $\lambda_h^*$  such that  $\Lambda_h = (0, \lambda_h^*)$  or  $\Lambda_h = (0, \lambda_h^*]$ .*

The idea of the proof is to show that if  $A$  is an asymptotic basis of order  $h$  with eigenvalue  $\lambda$ , and if  $0 < \lambda' < \lambda$ , then one can adjoin nonnegative integers to the set  $A$  to obtain an asymptotic basis of order  $h$  with eigenvalue  $\lambda'$ . Thus,  $\Lambda_h$  is an interval. Combinatorial and geometric arguments show that the additive spectrum is bounded above.



**Problem 15.** *Compute the upper bound  $\lambda_h^*$  of the additive spectrum  $\Lambda_h$ . Is this upper bound an eigenvalue?*

## 8. MINIMAL ASYMPTOTIC BASES

The set  $A$  of nonnegative integers is an asymptotic basis of order  $h$  if every sufficiently large integer is the sum of exactly  $h$  elements of  $A$ .

**Definition 12.** *An asymptotic basis  $A$  of order  $h$  is minimal if, for every element  $a^* \in A$ , the set  $A \setminus \{a^*\}$  is not an asymptotic basis of order  $h$ .*

Thus, if  $A$  is a minimal asymptotic basis of order  $h$ , then for every integer  $a^* \in A$  there are infinitely many positive integers  $n$  that cannot be represented as the sum of  $h$  elements of the set  $A \setminus \{a^*\}$ . Equivalently, every element of  $A$  is somehow “responsible” for the representation of infinitely many integers.

**Theorem 14** (Härtter [11], Nathanson [22]). *For every  $h \geq 2$  there exist minimal asymptotic bases of order  $h$ .*

On the other hand, it is not true that every asymptotic basis of order  $h$  contains a subset that is a minimal asymptotic basis of order  $h$ . In particular, we have the following result.

**Theorem 15** (Erdős-Nathanson [6]). *There exists an asymptotic basis  $A$  of order 2 such that, for every subset  $S \subseteq A$ , the set  $A \setminus S$  is an asymptotic basis of order 2 if and only if  $S$  is finite.*

Since there is no maximal finite subset of an infinite set, it follows that there exists an asymptotic basis of order 2 that contains no minimal asymptotic basis of order 2.

**Problem 16.** *Let  $h \geq 3$ . Construct an asymptotic basis  $A$  of order  $h$  such that, for every subset  $S \subseteq A$ , the set  $A \setminus S$  is an asymptotic basis of order  $h$  if and only if  $S$  is finite.*

**Problem 17.** *Find necessary and sufficient conditions to determine if an asymptotic basis  $A$  of order  $h$  contains a minimal asymptotic basis of order  $h$ .*

In a minimal asymptotic basis every element in the basis is responsible for the representation of infinitely many numbers. In particular, if  $A$  is a minimal asymptotic basis of order 2, then there must be infinitely many positive integers with a unique representation as the sum of two elements of  $A$ .

Let  $A$  be a set of integers. Let  $r_{A,2}(n)$  denote the *unordered representation function* of the set  $A$ , that is,

$$r_{A,2}(n) = |\{\{a_i, a_j\} \subseteq A : n = a_i + a_j\}|.$$

**Theorem 16** (Erdős-Nathanson [8]). *Let  $A$  be a set of nonnegative integers. If  $r_{A,2}(n) > c \log n$  for some  $c > 1/\log(4/3)$  and all  $n \geq n_0$ , then  $A$  contains a minimal asymptotic basis of order 2.*

**Problem 18.** *Is this true if  $r_{A,2}(n) > c \log n$  for some  $c > 0$ ?*

**Problem 19.** *Let  $A$  be a set of nonnegative integers. If  $r_{A,2}(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , does  $A$  contain a minimal asymptotic basis of order 2?*

The idea of minimal asymptotic basis can be generalized in the following way.

**Definition 13.** Let  $r \geq 1$ . The set  $A$  is an  $r$ -minimal asymptotic basis of order  $h$  if, for every  $S \subseteq A$ , the set  $A \setminus S$  is an asymptotic basis of order  $h$  if and only if  $|S| < r$ .

**Theorem 17** (Erdős-Nathanson [6]). For every  $r \geq 1$ , there exist  $r$ -minimal asymptotic bases of order 2.

**Problem 20.** Let  $h \geq 3$  and  $r \geq 2$ . Construct an  $r$ -minimal asymptotic basis  $A$  of order  $h$ .

## 9. MAXIMAL ASYMPTOTIC NONBASES

Maximal asymptotic nonbases are the natural dual to minimal asymptotic bases.

**Definition 14.** The set  $A$  of nonnegative integers is an asymptotic nonbasis of order  $h$  if it is not an asymptotic basis of order  $h$ , that is, if  $hA$  omits infinitely many nonnegative integers, that is, the set  $\mathbf{N} \setminus hA$  is infinite.

**Definition 15.** The set  $A$  of nonnegative integers is a maximal asymptotic nonbasis of order  $h$  if  $A$  is an asymptotic nonbasis of order  $h$  such that, for every integer  $a^* \in \mathbf{N} \setminus A$ , the set  $A \cup \{a^*\}$  is an asymptotic basis of order  $h$ .

The construction of minimal asymptotic bases is difficult, but it is easy to find simple examples of maximal nonbases. For example, the set of all nonnegative even integers is a maximal asymptotic nonbasis of order  $h$  for all  $h \geq 2$ . The set of all nonnegative multiples of a fixed prime number is a maximal asymptotic nonbasis of order  $p$ . Other examples can be constructed by taking appropriate unions of congruence classes.

**Theorem 18** (Nathanson [23]). There exist maximal asymptotic nonbases of zero asymptotic density.

The follow result implies that there exist asymptotic nonbases that cannot be embedded in maximal asymptotic nonbases.

**Theorem 19** (Hennefeld [14]). There exists an asymptotic nonbasis of order 2 such that, for every set  $S \subseteq \mathbf{N} \setminus A$ , the set  $A \cup S$  is an asymptotic nonbasis of order 2 if and only if  $|\mathbf{N} \setminus (A \cup S)|$  is infinite.

There are many beautiful results on minimal asymptotic bases and maximal asymptotic nonbases. Here are two of my favorites.

**Theorem 20** (Erdős-Nathanson [7]). There exists a partition of  $\mathbf{N}$  into two sets  $A$  and  $B$  such that  $A$  is a minimal asymptotic basis of order 2 and  $B$  is a maximal asymptotic nonbasis of order 2.

**Theorem 21** (Erdős-Nathanson [7]). There exists a partition of  $\mathbf{N}$  into two sets  $A$  and  $B$  such that, for any finite subset  $F$  of  $A$  and any finite subset  $G$  of  $B$ , the partition of  $\mathbf{N}$  into the sets

$$(A \setminus F) \cup G$$

and

$$(B \setminus G) \cup F$$

has the following property:

- (i) If  $|F| = |G|$ , then  $(A \setminus F) \cup G$  is a minimal asymptotic basis of order 2 and  $(B \setminus G) \cup F$  is a maximal asymptotic nonbasis of order 2.

- (ii) If  $|F| = |G| + 1$ , then  $(A \setminus F) \cup G$  is a maximal asymptotic nonbasis of order 2 and  $(B \setminus G) \cup F$  is a minimal asymptotic basis of order 2.

#### 10. COMPLEMENTING SETS OF INTEGERS

Let  $A$  and  $B$  be sets of integers, or, more generally, subsets of any additive abelian semigroup  $\mathbf{X}$ , and let  $A + B = \{a + b : a \in A, b \in B\} = C$ . If every element of the sumset  $C$  has a *unique* representation as the sum of an element of  $A$  and an element of  $B$ , then we write  $A \oplus B = C$ . We say that the set  $A$  *tessellates* the semigroup  $\mathbf{X}$  if there exists a set  $B$  such that  $A \oplus B = \mathbf{X}$ , and that  $A$  and  $B$  are *complementing subsets of  $\mathbf{X}$* .

In this section we consider complementing sets of integers. We examine the special case when  $A$  is a finite set of integers, and we want to determine if there exists an infinite set  $B$  of integers such that  $A \oplus B = \mathbf{Z}$ . By translation, we can always assume that  $A$  is a finite set of integers with  $0 \in A$ , and that  $0$  also belongs to  $B$ .

We call a set  $B$  *periodic* with *period*  $m$  if  $b \in B$  implies that  $b \pm m \in B$ .

**Theorem 22** (D. J. Newmann [33]). *Let  $A$  be a finite set of integers. If there exists a set  $B$  such that  $A \oplus B = \mathbf{Z}$ , then  $B$  is periodic with period*

$$m \leq 2^{\text{diam}(A)}$$

where  $\text{diam}(A) = \max(A) - \min(A)$ .

It follows that if  $A \oplus B = \mathbf{Z}$ , then  $B$  is a union of congruence classes modulo  $m$ . Defining  $\bar{A} = \{a + m\mathbf{Z} : a \in A\}$  and  $\bar{B} = \{b + m\mathbf{Z} : b \in B\}$ , we obtain a complementing pair  $\bar{A} \oplus \bar{B} = \mathbf{Z}/m\mathbf{Z}$ . Conversely, suppose that  $\bar{A}$  and  $\bar{B}$  are sets of congruence classes modulo  $m$  such that  $\bar{A} \oplus \bar{B} = \mathbf{Z}/m\mathbf{Z}$ . Let  $A$  be a set of representatives of the congruence classes in  $\bar{A}$  and let  $B$  be the union of the congruence classes in  $\bar{B}$ . Then  $A \oplus B = \mathbf{Z}$ .

**Theorem 23** (Kolountzakis [17], Ruzsa [34, 35]). *Let  $A$  and  $B$  be sets of integers such that  $A$  is finite,  $A \oplus B = \mathbf{Z}$ , and  $B$  has minimal period  $m$ . Then*

$$m \ll e^{c\sqrt{\text{diam}(A)}}.$$

**Theorem 24** (Biro [1]). *Let  $A$  and  $B$  be sets of integers such that  $A$  is finite,  $A \oplus B = \mathbf{Z}$ , and  $B$  has minimal period  $m$ . Then*

$$m \ll e^{c\sqrt[3]{\text{diam}(A)}}.$$

**Problem 21.** *Find the least upper bound for the period of a set  $B$  of integers that is complementary to a finite set  $A$  of diameter  $d$ .*

We can generalize the problem of complementing sets of integers to higher dimensions. Let  $d \geq 2$  and let  $A$  be a finite set of lattice points in  $\mathbf{Z}^d$ . Suppose there exists a set  $B \subseteq \mathbf{Z}^d$  such that  $A \oplus B = \mathbf{Z}^d$ . The following problem is well-known.

**Problem 22.** *Is the set  $B$  periodic even in one direction? Equivalently, does there exist a lattice point  $b_0 \in \mathbf{Z}^d \setminus \{0\}$  such that  $B + \{b_0\} = B$ ?*

We can also generalize the problem of complementing sets of integers to linear forms. Rewrite the original question as follows: Let  $\varphi(x, y) = x + y$ . For sets  $A, B \subseteq \mathbf{Z}$ , we define the set

$$\varphi(A, B) = \{\varphi(a, b) : a \in A, b \in B\}$$

and the representation function

$$r_{A,B}^{(\varphi)}(n) = \{(a, b) \in A \times B : \varphi(a, b) = n\}.$$

Given a finite set  $A$ , does there exist a set  $B$  such that  $\varphi(A, B) = \mathbf{Z}$  and  $r_{A,B}^{(\varphi)}(n) = 1$  for all integers  $n$ ? Now consider the linear forms

$$\psi(x_1, \dots, x_h) = u_1x_1 + \dots + u_hx_h$$

and

$$\varphi(x_1, \dots, x_h, y) = \psi(x_1, \dots, x_h) + vy = u_1x_1 + \dots + u_hx_h + vy$$

with nonzero integer coefficients  $u_1, \dots, u_h, v$ . Given an  $h$ -tuple  $\mathcal{A} = (A_1, \dots, A_h)$  of finite sets of integers, and a set  $B$  of integers, we define the set

$$\varphi(\mathcal{A}, B) = \{u_1a_1 + \dots + u_ha_h + vb : a_i \in A_i \text{ for } i = 1, \dots, h \text{ and } b \in B\}$$

and the representation function

$$r_{\mathcal{A},B}^{(\varphi)}(n) = \{(a_1, \dots, a_h, b) \in A_1 \times \dots \times A_h \times B : \varphi(a_1, \dots, a_h, b) = n\}.$$

**Problem 23.** *Given an  $h$ -tuple  $\mathcal{A} = (A_1, \dots, A_h)$  of finite sets of integers, determine if there exists a set  $B$  such that  $\varphi(\mathcal{A}, B) = \mathbf{Z}$  and  $r_{\mathcal{A},B}^{(\varphi)}(n) = 1$  for all integers  $n$ ?*

In this case, we say that  $\mathcal{A}$  and  $B$  are complementing sets of integers with respect to the linear form  $\varphi$ .

**Theorem 25** (Nathanson [30]). *If  $\mathcal{A}$  and  $B$  are complementing sets of integers with respect to the linear form  $\varphi$ , then  $B$  is periodic with period*

$$m \leq 2^{\frac{\text{diam}(\psi(A_1, \dots, A_h))}{|v|}}.$$

Ljubic and Nathanson [18] have extended Biro's cube root upper bound for the period of  $m$  to complementing sets of integers with respect to a linear form.

Instead of considering only sets that produce a unique representation for every integer, we can ask for any prescribed number of representations. This suggests the following inverse problem for representation functions associated to linear forms:

**Problem 24.** *Let  $\varphi(x_1, \dots, x_h, y) = u_1x_1 + \dots + u_hx_h + vy$  be a linear form with integer coefficients, and let  $\mathcal{A} = (A_1, \dots, A_h)$  be an  $h$ -tuple of finite sets of integers. Given a function  $f : \mathbf{Z} \rightarrow \mathbf{N}$ , does there exist a set  $B \subseteq \mathbf{Z}$  such that  $r_{\mathcal{A},B}^{(\varphi)}(n) = f(n)$  for all integers  $n$ ?*

We have the following compactness theorem.

**Theorem 26** (Nathanson [30]). *Let  $\varphi(x_1, \dots, x_h, y) = u_1x_1 + \dots + u_hx_h + vy$  be a linear form with integer coefficients, and let  $\mathcal{A} = (A_1, \dots, A_h)$  be an  $h$ -tuple of finite sets of integers. Consider a function  $f : \mathbf{Z} \rightarrow \mathbf{N}$ . Suppose there exists a strictly increasing sequence  $\{k_i\}_{i=1}^{\infty}$  of positive integers and a sequence  $\{B_i\}_{i=1}^{\infty}$  of (not necessarily increasing) sets of integers such that  $r_{\mathcal{A},B_i}^{(\varphi)}(n) = f(n)$  for integers  $n$  satisfying  $|n| \leq k_i$ . Then there exists an infinite set  $B$  such that  $r_{\mathcal{A},B}^{(\varphi)}(n) = f(n)$  for all integers  $n$ .*

## 11. THE CACCETTA-HÄGGKVIST CONJECTURE

Let  $G = G(V, E)$  be a finite directed graph with vertex set  $V$  and edge set  $E$ . Let  $n = |V|$ . Every edge  $e \in E$  is an ordered pair  $(v, v')$  of vertices. The vertex  $v$  is called the *tail* of  $e$  and the vertex  $v'$  is called the *head* of  $e$ . An edge of the form  $(v, v)$  is called a *loop*. We consider only graphs that may have loops, but that do not have multiple edges. A *path of length  $r$*  in the graph  $G$  is a finite sequence of edges  $e_1, e_2, \dots, e_r$ , where  $e_i = (v_i, v'_i)$  for  $i = 1, \dots, r$ , and  $v'_i = v_{i+1}$  for  $i = 1, \dots, r-1$ . The path is called a *circuit* if  $v'_r = v_1$ . A circuit of length 1 is a loop, a circuit of length 2 is called a *digon*, and a circuit of length 3 is called a *triangle*.

The *outdegree* of a vertex  $v$ , denoted  $\text{outdegree}(v)$ , is the number of edges  $e \in E$  whose tail is  $v$ . If  $|V| = n$  and  $\text{outdegree}(v) \geq 1$  for every vertex  $v \in V$ , then the graph  $G$  contains a circuit of length at most  $n$ . If  $|V| = n$  and  $\text{outdegree}(v) \geq 2$  for every vertex  $v \in V$ , then it is known that the graph  $G$  contains a circuit of length at most  $n/2$ .

**Conjecture 2** (Caccetta-Häggkvist). *Let  $k \geq 3$ . If  $\text{outdegree}(v) \geq k$  for every vertex  $v \in V$ , then the graph  $G$  contains a circuit of length at most  $n/k$ . Equivalently, if  $\text{outdeg}(v) \geq n/k$  for every vertex  $v \in V$ , then the graph  $G$  contains a circuit of length at most  $k$ .*

Even the case  $k = 3$  of the Caccetta-Häggkvist is open: If  $G$  is a graph with  $n$  vertices and if every vertex is the tail of at least  $n/3$  edges, prove that the graph contains a loop, a digon, or a directed triangle. This is a fundamental unsolved problem in graph theory.

**Definition 16.** *Let  $\Gamma$  be a finite group and let  $X \subseteq \Gamma$ . The Cayley graph  $G(V, E)$  is the graph with vertex set  $V = \Gamma$  and edge set  $E := \{(\gamma, \gamma x) : \gamma \in \Gamma, x \in X\}$ .*

**Theorem 27** (Hamidoune [9, 10]). *The Caccetta-Häggkvist conjecture is true for all Cayley graphs and for all vertex-transitive graphs..*

One proof of this result uses a theorem of Kemperman [16] in additive number theory. An exposition of this and other related results appears in Nathanson [27].

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